

# Open quantum systems with loss and gain

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## Abstract

We consider different properties of small open quantum systems coupled to an environment and described by a non-Hermitian Hamilton operator. Of special interest is the non-analytical behavior of the eigenvalues in the vicinity of singular points, the so-called exceptional points (EPs), at which the eigenvalues of two states coalesce and the corresponding eigenfunctions are linearly dependent from one another. The phases of the eigenfunctions are not rigid in approaching an EP and providing therewith the possibility to put information from the environment into the system. All characteristic properties of non-Hermitian quantum systems hold true not only for natural open quantum systems that suffer *loss* due to their embedding into the continuum of scattering wavefunctions. They appear also in systems coupled to different layers some of which provide *gain* to the system. Thereby gain and loss, respectively, may be fixed inside every layer, i.e. characteristic of it.

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## I. INTRODUCTION

During last about 15 years, quantum systems described by a non-Hermitian Hamilton operator with  $\mathcal{PT}$  symmetry and entirely real eigenvalues in a broad parameter range are considered intensely in literature, e.g. [1–3]. According to theory, the eigenvalues of the non-Hermitian Hamiltonian become complex if  $\mathcal{PT}$  symmetry is broken. Using the formal equivalence of the quantum mechanical Schrödinger equation to the optical wave equation in  $\mathcal{PT}$ -symmetric optical lattices [4–7], this theoretical result has been tested successfully about 5 years ago [8, 9], see also [10].

Already in 2010 the relation between  $\mathcal{PT}$ -symmetry breaking and the existence of exceptional points (EPs) in quantum systems, that are described by a non-Hermitian operator, is discussed [11]. The EP is, according to the definition by Kato [12], a singular point. In an open quantum system, it appears in the continuum of scattering wavefunctions into which the considered (local) system is embedded. Here, two eigenvalues of the non-Hermitian Hamiltonian coalesce and the corresponding two eigenfunctions are linearly dependent from one another, see e.g. the review [13]. As a consequence, the eigenvalues show a non-analytical behavior in the neighborhood of an EP and the two eigenfunctions are mixed strongly (entangled) in a finite parameter range around an EP. The role of EPs for  $\mathcal{PT}$ -symmetry breaking has been discussed also in other papers, see [3].

An open quantum system is described, in a very natural manner, by a non-Hermitian Hamilton operator  $\mathcal{H}$  the complex eigenvalues  $\mathcal{E}_k = E_k + \frac{i}{2}\Gamma_k$  of which provide not only the energies  $E_k$  of the states but also their lifetimes which are inverse proportional to the widths  $\Gamma_k$ . The eigenvalues and eigenfunctions contain the feedback from the environment of scattering wavefunctions onto the system properties. At low level density, the feedback can be neglected to a good approximation, and the numerical results obtained by using  $\mathcal{H}$  agree well with those of the standard Hermitian quantum physics. At high level density, however, the feedback cannot be neglected and the eigenvalues and eigenfunctions of  $\mathcal{H}$  may differ dramatically from those obtained from a Hermitian Hamiltonian. The results being often counterintuitive, appear not only in theoretical studies. Quite the contrary, mostly they are obtained initially in experimental studies. They are shown to be caused by EPs, see e.g. the review [13] and references therein. The results of further calculations on the basis of a schematical model show that an EP influences not only the eigenvalues and eigenfunctions

of a non-Hermitian Hamiltonian in a finite parameter range around its position but is itself also influenced by another nearby state. As a consequence, the system achieves a dynamical phase transition, i.e. the eigenstates of  $\mathcal{H}$  lose their spectral relation to the original states of the system (at low level density) [37].

The question arises therefore whether or not the results observed experimentally in [8–10] are characteristic of  $\mathcal{PT}$  symmetry and its breaking or are they solely a property related to the non-Hermiticity of the Hamiltonian. In the present paper we try to find an answer to this question. To begin with, we provide in Sect. II the non-Hermitian Hamiltonian  $\mathcal{H}^{(2)}$  of an open quantum system, by restricting to altogether two states, and discuss its eigenvalues and eigenfunctions in the following Sects. III and IV, respectively. In Sect. V the question is considered how much information can be extracted from a study of the cross section ( $S$ -matrix). In the following Sect. VI, the meaning of an imaginary coupling term between the states of an open system and its environment is discussed and, eventually, gain is included in the description of an open quantum system in Sect. VII. Some conclusions are drawn in the last section.

## II. HAMILTONIAN OF THE NATURAL OPEN QUANTUM SYSTEM

In an open quantum system, the discrete states described by a Hermitian Hamiltonian  $H^B$ , are embedded into the continuum of scattering wavefunctions, which exists always and can not be deleted. Due to this fact the discrete states turn into resonance states the lifetime of which is usually finite. The Hamiltonian  $\mathcal{H}$  of the open quantum system reads [13]

$$\mathcal{H} = H^B + V_{BC}G_C^{(+)}V_{CB} \quad (1)$$

where  $V_{BC}$  and  $V_{CB}$  stand for the interaction between system and environment and  $G_C^{(+)}$  is the Green function in the environment. The so-called internal (first-order) interaction between two states  $i$  and  $j$  is involved in  $H^B$  while their external (second-order) interaction via the common environment is described by the last term of (1). The eigenvalues of  $\mathcal{H}$  are complex and provide not only the energies of the states but also their lifetimes (being inverse proportional to the widths).

Generally, the coupling matrix elements of the external interaction consist of the principal

value integral

$$\text{Re} \langle \Phi_i^B | \mathcal{H} | \Phi_j^B \rangle - E_i^B \delta_{ij} = \frac{1}{2\pi} \mathcal{P} \int_{\epsilon_c}^{\epsilon'_c} dE' \frac{\gamma_{ic}^0 \gamma_{jc}^0}{E - E'} \quad (2)$$

which is real, and the residuum

$$\text{Im} \langle \Phi_i^B | \mathcal{H} | \Phi_j^B \rangle = -\frac{1}{2} \gamma_{ic}^0 \gamma_{jc}^0 \quad (3)$$

which is imaginary [13]. Here, the  $\Phi_i^B$  and  $E_i^B$  are the eigenfunctions and (discrete) eigenvalues, respectively, of the Hermitian Hamiltonian  $H^B$  which describes the states in the subspace of discrete states without any coupling to the environment. The  $\gamma_{ic}^0 \equiv \sqrt{2\pi} \langle \Phi_i^B | V | \xi_c^E \rangle$  are the (energy-dependent) coupling matrix elements between the discrete states  $i$  of the system and the environment of scattering wavefunctions  $\xi_c^E$ . The  $\gamma_{ic}^0$  have to be calculated for every state  $i$  and for each channel  $c$  (for details see [13]). When  $i = j$ , (2) and (3) give the selfenergy of the state  $i$ . The coupling matrix elements (2) and (3) (by adding  $E_i^B \delta_{ij}$  in the first case) are often simulated by complex values  $\omega_{ij}$ .

### III. EIGENVALUES OF THE NON-HERMITIAN HAMILTONIAN

In order to study the interaction of two states via the common environment it is convenient to start from two resonance states (instead of two discrete states). Let us consider, as an example, the symmetric  $2 \times 2$  matrix

$$\mathcal{H}^{(2)} = \begin{pmatrix} \varepsilon_1 \equiv e_1 + \frac{i}{2}\gamma_1 & \omega_{12} \\ \omega_{21} & \varepsilon_2 \equiv e_2 + \frac{i}{2}\gamma_2 \end{pmatrix} \quad (4)$$

the diagonal elements of which are the two complex eigenvalues  $\varepsilon_i$  ( $i = 1, 2$ ) of a non-Hermitian operator  $\mathcal{H}^0$  (with  $\gamma_i \leq 0$  for the decay width of the state  $i$  of the open quantum system). The  $e_i$  and  $\gamma_i$  denote the energies and widths, respectively, of the two states when their interaction via the continuum vanishes,  $\omega_{ij} = 0$ . (Note that the width  $\gamma_i$  has the dimension of energy  $E$  while the dimension of the coupling matrix elements  $\gamma_{ic}^0$  defined in (2) and (3) is  $\sqrt{E}$  according to the definitions used usually in literature). The  $\omega_{12} = \omega_{21} \equiv \omega$  stand for the coupling of the two states via the common environment. The selfenergy of the states is assumed to be included into the  $\varepsilon_i$ .

The two eigenvalues of  $\mathcal{H}^{(2)}$  are

$$\mathcal{E}_{i,j} \equiv E_{i,j} + \frac{i}{2}\Gamma_{i,j} = \frac{\varepsilon_1 + \varepsilon_2}{2} \pm Z ; \quad Z \equiv \frac{1}{2} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 4\omega^2} \quad (5)$$

where  $E_i$  and  $\Gamma_i \leq 0$  stand for the energy and width, respectively, of the eigenstate  $i$ . Resonance states with nonvanishing widths  $\Gamma_i$  repel each other in energy according to the value of  $\text{Re}(Z)$  while the widths bifurcate according to the value of  $\text{Im}(Z)$ . The two states cross when  $Z = 0$ . This crossing point is a singular point called mostly exceptional point (EP) according to the definition of Kato [12]. Here, the two eigenvalues coalesce,  $\mathcal{E}_1 = \mathcal{E}_2$ , and the S matrix has a double pole [14–17]. Another notation for the EP is *branch point in the complex plane* [14–19]. In the neighborhood of an EP, the parameter dependence of the eigenvalues  $\mathcal{E}_i$  is non-analytical, especially that of the widths  $\Gamma_i$  (for illustration see Fig. 4 in [20]). That means, Fermi’s golden rule is violated under the influence of an EP.

The condition  $Z = 0$  cannot be fulfilled for any two discrete eigenvalues (with  $e_1 \neq e_2$ ) of the Hermitian operator  $H^B$ . This result is known in standard Hermitian quantum physics since many years: two discrete states avoid always crossing [21, 22]. At the critical parameter value of the avoided crossing, a geometrical phase, the Berry phase [23, 24], appears.

In difference to this, the condition  $Z = 0$  can generally be fulfilled for resonance states (when their widths  $\gamma_i$  are sufficiently large). This fact causes, on the one hand, differences between open and closed systems. Among others, the geometrical phase related to an EP in the two-level case, differs from the Berry phase by a factor 2, e.g. [13], such that the topology of an open system differs from that of a closed system. On the other hand, discrete states and resonance states show an analog behavior related to crossing: discrete and narrow resonance states avoid crossing while broad resonances avoid overlapping. At high level density where one naively would expect a strong overlapping, the resonances avoid each other: one resonance accumulates almost the whole sum of the widths of all resonances while the remaining ones become nearly stable as illustrated in [25] for altogether four resonances by means of the corresponding S-matrix poles. This example is an illustration of the fact that, in difference to discrete states, the resonances can avoid each other not only by means of level repulsion caused by  $\text{Re}(Z)$  but also by width bifurcation caused by  $\text{Im}(Z)$ . It is possible that the states cross in energy according to  $\text{Re}(Z) = 0$  while their widths bifurcate according to  $\text{Im}(Z) \neq 0$  (for details see [13]). Due to width bifurcation, even bound states in the continuum (with vanishing width  $\Gamma_i$ ) may appear, see e.g. [14, 15, 26] and references therein.

An additional remark should be added here: the condition  $Z = 0$  is fulfilled at only one point in the continuum, the EP, at the most. It is therefore of measure zero. However,

avoided resonance overlapping caused by it in its neighborhood determines the dynamics of open quantum systems, see [13].

#### IV. EIGENFUNCTIONS OF THE NON-HERMITIAN HAMILTONIAN

As discussed in the foregoing section III, an EP is defined by the coalescence of two eigenvalues,  $\mathcal{E}_1 = \mathcal{E}_2$ , of the non-Hermitian operator  $\mathcal{H}^{(2)}$ . The corresponding eigenfunctions  $\Phi_{1,2}$  of  $\mathcal{H}^{(2)}$  of the two crossing states are linearly dependent from one another at the EP,

$$\Phi_1^{\text{cr}} \rightarrow \pm i \Phi_2^{\text{cr}} ; \quad \Phi_2^{\text{cr}} \rightarrow \mp i \Phi_1^{\text{cr}} \quad (6)$$

according to analytical as well as numerical and experimental studies (for details and examples see [13, 27, 28]). The eigenfunctions  $\Phi_k$  of the non-Hermitian  $\mathcal{H}^{(2)}$  are biorthogonal because  $H|\Phi_k\rangle = \mathcal{E}_k|\Phi_k\rangle$  and  $\langle\Phi_k^*|H = \mathcal{E}_k\langle\Phi_k^*|$  (where  $\mathcal{E}_k$  is an eigenvalue of  $H$  and the vectors  $|\Phi_k\rangle$  and  $\langle\Phi_k^*|$  denote its right and left eigenfunctions, respectively). They have to be normalized therefore by means of the complex value  $\langle\Phi_k^*|\Phi_l\rangle$  what is in contrast to the normalization of the eigenfunctions of a Hermitian operator by means of the real value  $\langle\Phi_k|\Phi_l\rangle$  (for details see sections 2.2 and 2.3 of [13]). We use the normalization

$$\langle\Phi_k^*|\Phi_l\rangle = \delta_{kl} \quad (7)$$

in analogy to  $\langle\Phi_k|\Phi_l\rangle = \delta_{kl}$  for discrete states in order to describe a smooth transition from the closed system with discrete states and orthogonal eigenfunctions to the weakly open system with narrow resonance states and biorthogonal eigenfunctions. The  $\Phi_k$  contain (like the  $\mathcal{E}_k$ ) global features that are caused by many-body forces induced by the coupling  $\omega_{kl}$  of the two states  $k$  and  $l \neq k$  via the environment. Moreover, they contain the self-energy contributions of the states  $k$  due to their coupling  $\omega_{kk}$  to the environment.

The biorthogonality of the eigenfunctions  $\Phi_k$  of the non-Hermitian operator  $\mathcal{H}^{(2)}$  is quantitatively expressed by the ratio

$$r_k \equiv \frac{\langle\Phi_k^*|\Phi_k\rangle}{\langle\Phi_k|\Phi_k\rangle}. \quad (8)$$

Usually  $r_k \approx 1$  for decaying states which are well separated from other decaying states (according to the fact that Hermitian quantum physics is a good approach at low level density and small coupling strength to the environment). Here, the eigenfunctions are (almost) orthogonal,  $\langle\Phi_k^*|\Phi_k\rangle \approx \langle\Phi_k|\Phi_k\rangle = 1$ .

The situation changes however completely when an EP is approached due to the fact that the eigenfunctions of two crossing states are linearly dependent according to (6) and, by using the normalization condition (7),  $\langle \Phi_k | \Phi_k \rangle \rightarrow \infty$  at the EP [13]. Thus, the phases of the two eigenfunctions relative to one another change dramatically when the crossing point is approached and  $r_k \rightarrow 0$ . The non-rigidity  $r_k$  of the phases of the eigenfunctions of  $\mathcal{H}^{(2)}$  follows, eventually, from the fact that  $\langle \Phi_k^* | \Phi_k \rangle$  is a complex number (in difference to the norm  $\langle \Phi_k | \Phi_k \rangle$  which is a real number) such that the normalization condition (7) can be fulfilled only by the additional postulation  $\text{Im}\langle \Phi_k^* | \Phi_k \rangle = 0$  (what corresponds to a rotation).

The value  $r_k$ , defined by (8), is called *phase rigidity* of the eigenfunction  $\Phi_k$  [13]. Generally  $1 \geq r_k \geq 0$ . When  $r_k < 1$ , an analytical expression for the eigenfunctions as function of a certain control parameter can, generally, not be obtained. The non-rigidity  $r_k < 1$  of the phases of the eigenfunctions of  $\mathcal{H}^{(2)}$  in the neighborhood of EPs is the most important difference between the non-Hermitian quantum physics and the Hermitian one. Mathematically, it causes nonlinear effects in quantum systems in a natural manner [13]. Physically, it gives one of the states of the system the possibility to align near to (and at) the EP with the common environment and to receive thereby a large width  $\Gamma_k$ . This alignment is nothing but a quantitative measure of the influence of the environment onto the spectroscopic properties of the system [13]. The aligned state is the short-lived state caused by avoided resonance overlapping. Its formation is accompanied by trapping the remaining resonance states which are long-lived, i.e. decoupled more or less from the environment.

The relations (6) and  $r_k < 1$  in approaching an EP are seen in experimental results obtained for two resonance states in microwave billiards [29, 30], see [13, 27]. In [28], wave transport in an open non-Hermitian quantum dot with  $\mathcal{PT}$  symmetry is calculated. The numerical results show clearly not only the relations (6) in approaching an EP but also the phase rotation, which takes place in the vicinity of an EP and is described by  $r_k < 1$  as stated above.

In order to receive a statement on the entanglement of the wavefunctions, it is meaningful to represent the eigenfunctions  $\Phi_i$  of  $\mathcal{H}^{(2)}$  in the set of basic wavefunctions  $\Phi_k^0$  of  $\mathcal{H}^0$

$$\Phi_k = \sum_{l=1}^N b_{kl} \Phi_l^0 ; \quad b_{kl} = |b_{kl}| e^{i\theta_{kl}} . \quad (9)$$

Also the  $b_{kl}$  are normalized according to the biorthogonality relations of the wavefunctions  $\{\Phi_k\}$ . The angle  $\theta_{kl}$  can be determined from  $\text{tg}(\theta_{kl}) = \text{Im}(b_{kl})/\text{Re}(b_{kl})$ . The entanglement

of the wavefunctions is large in the neighborhood of EPs as numerical calculations have shown, e.g. [31, 32].

## V. THE $S$ -MATRIX IN THE VICINITY OF AN EXCEPTIONAL POINT

The cross section can be calculated by means of the  $S$ -matrix  $\sigma(E) \propto |1 - S(E)|^2$ . In the vicinity of a single level coupled to one channel the line shape is the well-known Breit-Wigner shape,

$$S = 1 + i \frac{\Gamma_k}{E - E_k - \frac{i}{2}\Gamma_k} \quad (10)$$

where  $E$  is the energy and  $E_k$  and  $\Gamma_k$  are defined in Eq. (5). This expression can be rewritten as [33]

$$S = \frac{E - E_1 + \frac{i}{2}\Gamma_1}{E - E_1 - \frac{i}{2}\Gamma_1} \quad (11)$$

which is explicitly unitary. Extending the problem to that of two closely neighboring resonance states that are coupled to a common continuum, the representation (11) of the  $S$ -matrix reads (up to a background term)

$$S = \frac{(E - E_1 + \frac{i}{2}\Gamma_1)(E - E_2 + \frac{i}{2}\Gamma_2)}{(E - E_1 - \frac{i}{2}\Gamma_1)(E - E_2 - \frac{i}{2}\Gamma_2)}. \quad (12)$$

At an EP, the  $S$ -matrix has a double pole. Here (12) can be rewritten as [33],

$$S = 1 + 2i \frac{\Gamma_d}{E - E_d - \frac{i}{2}\Gamma_d} - \frac{\Gamma_d^2}{(E - E_d - \frac{i}{2}\Gamma_d)^2} \quad (13)$$

where  $E_1 = E_2 \equiv E_d$  and  $\Gamma_1 = \Gamma_2 \equiv \Gamma_d$ . The second term on the right-hand side of this expression corresponds to the usual linear term (10) describing a single-state Breit-Wigner resonance, however multiplied by a factor two. The third term is quadratic in energy. In the cross section, an interference minimum appears at the EP and the two peaks at both sides are asymmetric [32, 34]. The interference with a direct scattering (background) part may change the picture when the scattering phase differs from zero [35].

It is interesting to trace the line shape of a resonance when the distance to another resonance is varied [32]. According to (13) the line shape is of standard symmetrical Breit-Wigner form only when both resonances are well separated from one another. In approaching an EP, the two resonances avoid overlapping, the  $S$ -matrix approaches the expression (13) at



the EP and two asymmetric bumps (“resonances”) appear in the cross section. Finally, the cross section shows a broad resonance with a narrow dip in the center which is reminiscent of the long-lived resonance state caused by width bifurcation together with the short-lived (broad) resonance state by which it is superposed. Thus, the cross section varies smoothly in the whole parameter range including the critical region around the EP. It is difficult therefore to trace the influence of an EP on the dynamics of open quantum systems by considering only the cross section ( $S$ -matrix). In order to receive more information, the eigenvalues and eigenfunctions of the non-Hermitian operator should be considered which behave non-analytically at the EP (in spite of the smooth behavior of the  $S$ -matrix) and  $\langle \Phi_k | \Phi_k \rangle \rightarrow \infty$ , respectively.

## VI. SPECIAL CASE WITH IMAGINARY COUPLING

Generally, the expression  $Z$  defined in (5) is complex. By using the condition  $Z = 0$ , the EP can be found (when it is far from another EP [13]). Mostly the critical value  $\omega = \omega^{\text{cr}}$  is complex. In the limiting case with real  $\omega = \omega_r$ ,  $\gamma_1 = \gamma_2$  and  $e_1 \neq e_2$ , no EP exists because  $Z^2 > 0$ .

When however  $\omega = i\omega_i$  is imaginary,

$$Z = \frac{1}{2} \sqrt{(e_1 - e_2)^2 - \frac{1}{4}(\gamma_1 - \gamma_2)^2 + i(e_1 - e_2)(\gamma_1 - \gamma_2) - 4\omega_i^2}, \quad (14)$$

the condition  $Z = 0$  can be fulfilled when  $(e_1 - e_2)^2 - \frac{1}{4}(\gamma_1 - \gamma_2)^2 = 4\omega_i^2$  and  $(e_1 - e_2)(\gamma_1 - \gamma_2) = 0$ , i.e. when  $\gamma_1 = \gamma_2$  (or when  $e_1 = e_2$ ). Let  $e_i = e_i(a)$  be dependent on a certain parameter  $a$  and  $\omega = i\omega_i$  be fixed, then

$$(e_1(a) - e_2(a))^2 - 4\omega_i^2 = 0 \rightarrow e_1(a) - e_2(a) = \pm 2\omega_i \quad (15)$$

and two EPs appear. It holds further

$$(e_1(a) - e_2(a))^2 > 4\omega_i^2 \rightarrow Z \in \Re \quad (16)$$

$$(e_1(a) - e_2(a))^2 < 4\omega_i^2 \rightarrow Z \in \Im \quad (17)$$

independent of the parameter dependence of the  $e_i$ . In the first case, the eigenvalues  $\mathcal{E}_i = E_i + i/2 \Gamma_i$  differ from the original values  $\varepsilon_i = e_i + i/2 \gamma_i$  by a contribution to the energies and in the second case by a contribution to the widths. The width bifurcation starts in the

very neighborhood of one of the EPs and becomes maximum in the middle between the two EPs. This happens at the crossing point  $e_1 = e_2$  where  $\Delta\Gamma/2 \equiv |\Gamma_1/2 - \Gamma_2/2| = 4\omega_i$ . A similar situation appears when  $\gamma_1 \approx \gamma_2$  as results of numerical calculations show. For details see [32, 36].

## VII. INCLUSION OF GAIN

We start with the symmetric  $2 \times 2$  matrix (4) by assuming a different sign for  $\gamma_1$  and  $\gamma_2$  according to loss ( $\gamma_i \leq 0$ ) and gain ( $\gamma_i \geq 0$ ). The two eigenvalues of  $\mathcal{H}^{(2)}$  are given by (5) and  $Z$  reads

$$Z = \frac{1}{2} \sqrt{(e_1 - e_2)^2 - \frac{1}{4}(\gamma_1 - \gamma_2)^2 + i(e_1 - e_2)(\gamma_1 - \gamma_2) + 4\omega^2} \quad (18)$$

where  $\omega = \omega_r + i\omega_i$  is complex, generally. As for a natural open system, an EP appears when the condition  $Z = 0$  is fulfilled. However  $\gamma_1 \neq \gamma_2$  and therefore  $(\gamma_1 - \gamma_2)^2 \neq 0$  even when  $|\gamma_1| = |\gamma_2|$ . The conditions read

$$(e_1 - e_2)^2 - \frac{1}{4}(\gamma_1 - \gamma_2)^2 = -4\omega_r^2 + 4\omega_i^2 \quad (19)$$

$$(e_1 - e_2)(\gamma_1 - \gamma_2) = -8\omega_r\omega_i \quad (20)$$

which can mostly be fulfilled also when both values  $\gamma_i$ , i.e. loss *and* gain, are parameter independent and only the energies  $e_i$  are parameter dependent (e.g.  $e_i = e_i(a)$  where  $a$  is a suitable parameter). Two examples with real and complex  $\omega$ , respectively, are shown in Fig. 1. In both cases, one EP can be seen.

Let us consider now the special case with parameter dependent coupling strengths  $\omega = \omega(a)$  and fixed (parameter independent)  $e_i$  and  $\gamma_i$ . If  $e_1 = e_2$  and  $\omega = \omega_r(a)$ , it follows from (18)

$$-\frac{1}{4}(\gamma_1 - \gamma_2)^2 + 4\omega_r(a)^2 = 0 \rightarrow \gamma_1 - \gamma_2 = \pm 4\omega_r(a) \quad (21)$$

and two EPs appear. If  $\gamma_1 = -\gamma_2 \equiv \gamma$  the condition (21) reads  $\gamma/2 = \pm\omega_r(a)$ . Further

$$\frac{1}{4}(\gamma_1 - \gamma_2)^2 = \gamma^2 > 4\omega_r(a)^2 \rightarrow Z \in \Im \quad (22)$$

$$\frac{1}{4}(\gamma_1 - \gamma_2)^2 = \gamma^2 < 4\omega_r(a)^2 \rightarrow Z \in \Re \quad (23)$$

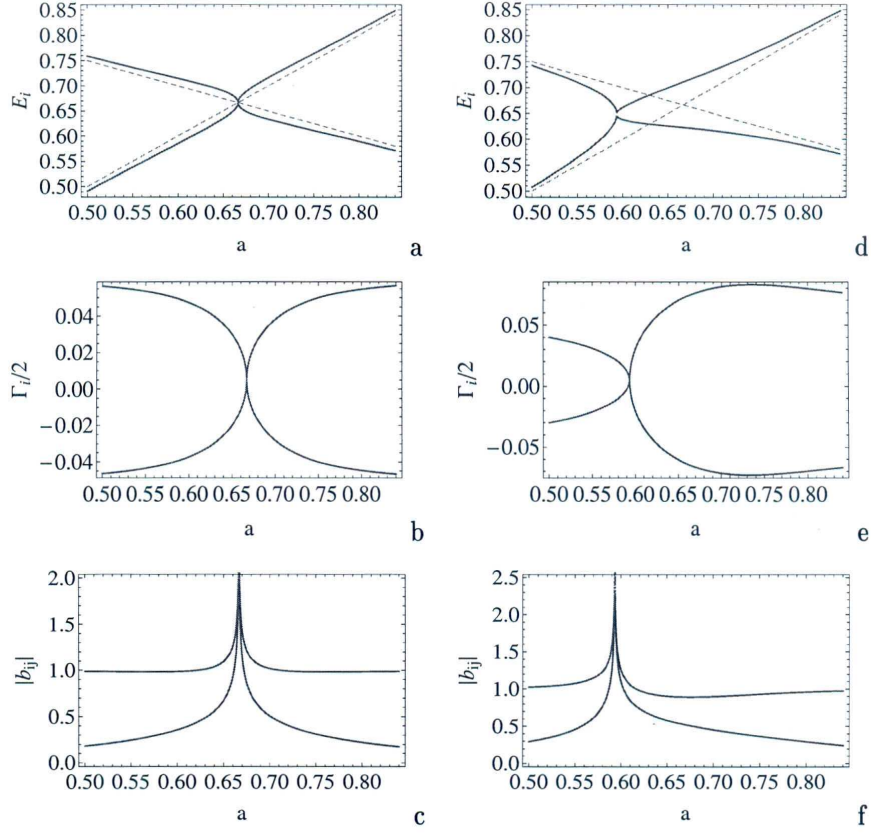


FIG. 1: Energies  $E_i$  (a, d), widths  $\Gamma_i/2$  (b, e) and mixing coefficients  $|b_{ij}|^2$  (c, f) of  $N = 2$  states coupled to  $K = 1$  channel as a function of the parameter  $a$ . The parameters are  $e_1 = 1 - a/2$ ,  $e_2 = a$ ;  $\gamma_1/2 = -0.05$ ;  $\gamma_2/2 = 0.06$ ;  $\omega = 0.055$  (left panel);  $\omega = 0.0789(1 + i)/\sqrt{2}$  (right panel). The dashed lines show  $e_i(a)$ .

independent of any parameter dependence of the  $e_i$  and  $\gamma_i$ . In the first case, the eigenvalues  $\mathcal{E}_i = E_i + i/2 \Gamma_i$  differ from the original values  $\varepsilon_i = e_i + i/2 \gamma_i$  by a contribution to the widths and in the second case by a contribution to the energies. The width bifurcation starts in the very neighborhood of one of the EPs and becomes maximum in the middle between the two EPs. Numerical results support this picture (Fig. 2 left panel). According to results of further numerical calculations, a similar situation appears when  $\gamma_1 \approx \gamma_2$  and (or)  $e_1 \approx e_2$ .

When  $\omega = i\omega_i(a)$ , the condition  $Z = 0$  cannot be fulfilled in the considered special case since

$$\frac{1}{4}(\gamma_1 - \gamma_2)^2 + 4\omega_i(a)^2 = \gamma^2 + 4\omega_i(a)^2 > 0 \quad (24)$$

and no EP exists. This scenario is analogue to that obtained for a natural open quantum

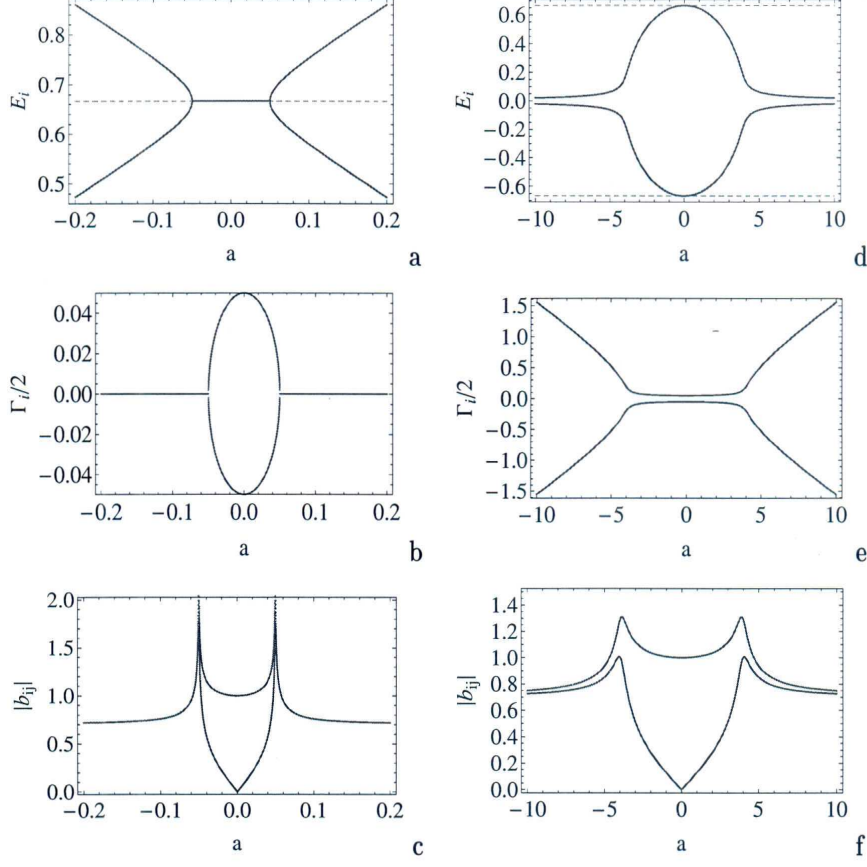


FIG. 2: Energies  $E_i$  (a, d), widths  $\Gamma_i/2$  (b, e) and mixing coefficients  $|b_{ij}|^2$  (c, f) of  $N = 2$  states coupled to  $K = 1$  channel as a function of the parameter  $a$ . The parameters are  $e_1 = 2/3$ ,  $e_2 = 2/3$ ;  $\gamma_1/2 = -0.05$ ;  $\gamma_2/2 = 0.05$ ;  $\omega = a$  (left panel);  $e_1 = 2/3$ ,  $e_2 = -2/3$ ;  $\gamma_1/2 = -0.05$ ;  $\gamma_2/2 = 0.05$ ;  $\omega = i \cdot a$  (right panel).

system when  $\omega$  is real, see Sect. VI. This correspondence is in agreement with Eqs. (2) and (3), according to which  $\omega$  is complex (almost imaginary) in the first case and real in the second case.

As a result, we have two EPs in an open system with two decaying states ( $\gamma_1 = \gamma_2 \neq 0$ ) when  $\omega = i\omega_i$  is imaginary. In contrast, there is no EP when  $\omega = \omega_r$  is real. The situation in an open system with gain and loss ( $\gamma_1 = -\gamma_2 \neq 0$ ) is the opposite way around: there is no EP when  $\omega = i\omega_i$  is imaginary, and two EPs when  $\omega = \omega_r$  is real. In both cases with two EPs we have width bifurcation between the two EPs. In the more realistic case with complex  $\omega$ , the results are similar to those discussed above when  $\omega_i \gg \omega_r$  and  $\omega_i \ll \omega_r$ , respectively.

Interesting are the results shown in Fig. 2 right panel. Here  $e_2 = -e_1$  and  $\gamma_2 = -\gamma_1$  are parameter independent while  $\omega(a) = i\omega_i(a)$  is imaginary and parameter dependent. The results are similar to those for a  $\mathcal{PT}$  symmetric system with parameter dependent  $\gamma_1(a') = -\gamma_2(a')$ , fixed  $e_1 = e_2$  and fixed real  $\omega$  (see Fig. 1 left panel in [36] where  $\Gamma_i = 0$  in the finite parameter range between the two EPs according to  $\mathcal{PT}$  symmetry and its breaking at the EPs). In order to see the EPs and  $\Gamma_i = 0$  in the parameter range between them also in the present case,  $\omega$  has however to be complex because (24) holds true when  $\omega = i\omega_i$ . The two critical values  $\omega_i(a^{\text{cr}})$  and  $\omega_r(a^{\text{cr}})$  at the EPs can be determined by solving the two equations (19) and (20) with the fixed values  $e_i$  and  $\gamma_i$ . The numerical results support the analytical ones. The results are however sensitive relative to small variations of the parameter  $a$ . In this manner, Fig. 2 illustrates that different situations with gain and loss can be realized in open quantum systems. A few of them are sensitive relative to small variations of  $a$  (e.g. Fig. 2 right panel) while most of them are more stable (e.g. Fig. 2 left panel).

It should be underlined here the following. In difference to a  $\mathcal{PT}$  symmetric system (see Fig. 1 left panel in [36]) the relations (21) to (23) are obtained by keeping fixed the widths  $\gamma_i$  (and the energies  $e_i$ ), and varying the coupling strengths  $\omega = \omega(a)$  by means of the parameter  $a$ . This condition is, probably, easier to realize experimentally than the parametrical dependence  $\gamma_i(a')$  of the widths. Remarkably is the possibility to realize very different situations in an open quantum system (compare left and right panels of Fig. 2).

In any case, the mathematical condition for an EP to occur in an open quantum system with loss and gain, is the same as that for a natural open quantum system, namely  $Z = 0$ . The only difference is that the two widths  $\gamma_i$  have different sign in the first case while they have the same sign in the last case. In both cases, the EP influences a finite parameter range in its neighborhood.

## VIII. CONCLUSIONS

In the present paper we have considered generic properties of open quantum systems which are embedded, in a natural manner, in the continuum of decay channels. Due to the coupling of the system states to the environment, the Hamiltonian  $\mathcal{H}$  describing the system is non-Hermitian. The eigenvalues  $\mathcal{E}_k$  of  $\mathcal{H}$  are complex, generally, and provide not only the

energies  $\text{Re}(\mathcal{E}_k)$  of the states but also their lifetimes being inverse proportional to the widths  $\text{Im}(\mathcal{E}_k)$ . Although the eigenfunctions  $\Phi_k$  of  $\mathcal{H}$  are biorthogonal, they can be normalized by  $\langle \Phi_k^* | \Phi_l \rangle = \delta_{kl}$  such that the transition from a weakly opened system with narrow resonance states to a strongly opened system with (at least) one short-lived resonance state occurs smoothly. This transition is possible because the phases of the eigenfunctions of  $\mathcal{H}$  are not rigid: while the wavefunctions of two states are (almost) orthogonal to one another at large distance from the EP (as those of the corresponding closed system), they become linearly dependent from one another in approaching the EP. In the vicinity of an EP, the environment is able to put information into the system by means of aligning one of the system states to the states of the environment and, at the same time, decoupling the remaining states from the environment. That means, a short-lived (the *aligned*) state appears together with long-lived (the decoupled, *trapped*) states, a process called mostly width bifurcation. The changes of the phase rigidity and of the lifetime of the states near to (or at) an EP are proven by Figs. 4 and 5 of [28] obtained for an open quantum billiard with  $\mathcal{PT}$  symmetry. Furthermore, the eigenfunctions of  $\mathcal{H}$  are strongly mixed (entangled) in a finite parameter range around an EP such that a nearby state will have a large influence not only in this parameter range but also somewhat beyond this range. A dynamical phase transition may therefore occur in the system which changes radically the spectroscopic properties of the system (a detailed discussion of the relation between EPs and dynamical phase transitions can be found in [37]). The dynamical phase transition has the same characteristic features which are known from  $\mathcal{PT}$ -symmetry breaking occurring in a  $\mathcal{PT}$ -symmetric system under some critical condition.

In the second part of the paper, we have studied the properties of open quantum systems by including “gain” from the environment. This is, of course, not realized in nature. As the results obtained some years ago for  $\mathcal{PT}$ -symmetric optical lattices [8, 9] and recent results for other systems show, it can however be realized in experiment, e.g. [38, 39]. The results of our theoretical studies presented in the present paper, indicate that the consideration of gain is not restricted to  $\mathcal{PT}$ -symmetric systems. It is a much more general phenomenon appearing in open quantum systems that are described by a non-Hermitian Hamiltonian. The conditions for desired properties can be formulated. In particular, it is possible to consider different layers with fixed (parameter independent) loss or gain which is characteristic of the layer in question. These results are of importance for basic research as well as for applications.

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